PD.TVZ.HR
POLYTECHNIC & DESIGN Vol. 12, No. 4, 2024.
Mrežna inačica: ISSN 2459-6302 DOI: 10.19279/TVZ.PD.2024-12-4-15
Tiskana inačica: ISSN 1849-1995

EULER AND PICARD METHODS FOR SOLVING FIRST-ORDER DIFFERENTIAL EQUATIONS – CLASSICAL AND SOFTWARE-BASED

Edmond Krusha[1,2], Marijan Čančarević[1], Ivica Čulina[3]

[1] *Veleučilište suvremenih informacijskih tehnologija u Zagrebu, Klaićeva 7, Zagreb, Hrvatska*
[2] *Tehničko veleučilište u Zagrebu, Vrbik 8, Zagreb, Hrvatska*
[3] *Veleučilište Velika Gorica, Zagrebačka 5,Velika Gorica, Hrvatska*

SAŽETAK

U ovom radu su opisani i na jednom primjeru izvedeni algoritmi Eulerove i Picardove metode za rješavanje diferencijalnih jednadžbi 1. reda. Obje metode daju aproksimativno rješenje, pri čemu Eulerova metoda [1], [3] daje numeričko rješenje a Picardova [2] najprije analitičko i potom po potrebi direktnim uvrštavanjem odgovarajuće numeričko rješenje. Na primjeru su pokazani navedeni algoritmi i to klasičnim načinom rješavanja i primjenom SageMath softwarea [4]. Temeljna svrha rada je ukazati na prednosti numeričkih metoda, posebno ukoliko se koriste suvremene tehnologije.

Ključne riječi: *numerička metoda, diferencijalna jednadžba, Euler, Picard, software, SageMath*

ABSTRACT

This paper describes and demonstrates the algorithms of Euler's and Picard's methods for solving first-order differential equations using a specific example. Both methods provide approximate solutions, with Euler's method [1], [3] yielding a numerical solution while Picard's method [2] initially provides an analytical solution, which can then be transformed into a numerical solution through direct substitution if necessary. The selected example illustrates these algorithms both through the classical solution approach and by utilizing the SageMath software [4]. The primary objective of this study is to highlight the advantages of numerical methods, particularly when modern technologies are employed.

Keywords: *numerical method, differential equation, Euler, Picard, software, SageMath*

1. UVOD
1. INTRODUCTION

Many processes and phenomena in physics, chemistry, biology, etc., are described using differential equations. For example, the rate of growth y' of a biological population as a function of time t is modeled by the differential equation $y'(t)=k\cdot y(t)$, where $y(t)$ represents the size of the biological population and k is the proportionality coefficient. In economics, marginal profit is derived from $D_m(p)=D'(p)$, where $D(p)$ represents the total profit as a function of product p, ... Often, differential equations cannot be solved using elementary analytical methods, or the path to a solution is lengthy and complicated. Therefore, it becomes almost essential to employ numerical methods for solving differential equations. This paper focuses on differential equations of the form $y'=f(x,y)$. The Euler's numerical method and the Picard's quasi-numerical method are introduced, along with their corresponding algorithms. Next, a specific differential equation was solved using an analytical approach algorithm, followed by applications of the Euler's method and Picard's method — both classically (manual computation) and through implementation in SageMath software.

2. EULEROVA METODA
2. EULER'S METHOD

Let's consider this differential equation

$$y' = f(x,y),\ y(x_0) = y_0 \tag{1}$$

and the real number $c > x_0$. The interval $[x_0,c]$

points $x_i = x_0 + i \cdot h,\ \ h = \frac{c - x_0}{n},\ i = 1,2,\ldots, n$ is divided into n equal parts. By using the geometrical meaning of the derivation, the equation of the tangent at a point of the graph is obtained $D_0\ (x_0, y_0)$:

$$y = y_0 + f(x_0, y_0) \cdot (x - x_0)$$

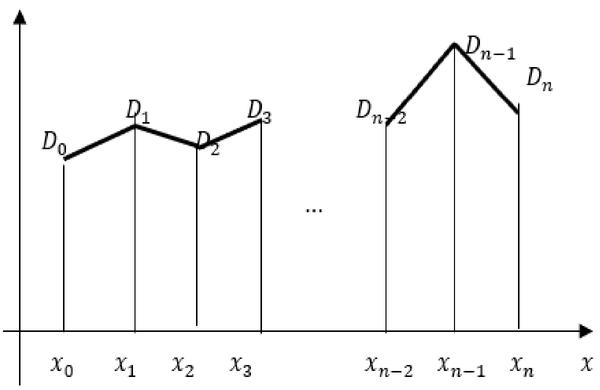
The intersection of the line $x = x_1$ and the tangent line provides

$$y_1 = y_0 + f(x_0, y_0) \cdot (x_1 - x_0) = \\ = y_0 + f(x_0, y_0)h.$$

Thus, the point $D_1\ (x_1, y_1)$ is obtained. By repeating the previous procedure for the point $D_1\ (x_1, y_1)$ and analogously for each subsequent xi through to x_n, a series of points is obtained $D_i\ (x_i, y_i)$. Those points determine the polygonal line that approximates the desired integral curve (Figure 1.). The ordinates y_i of the points $D_i\ (x_i, y_i)$ are the approximations of the solution to the differential equation over the interval $[x_0, c]$. It should be noted that the target result of the method is the value of the unknown function at the point $x_n = c$ and this result becomes more accurate as n increases. In simplified terms, the procedure generates a sequence of iterations

$$y_k = y_{k-1} + f(x_{k-1}, y_{k-1}) \cdot h, \\ k = 1,2, \ldots, n \qquad (2)$$

which facilitate finding the next approximation by way of the previous one.



***Slika 1** Određivanje poligonalne linije koja aproksimira traženu integralnu krivulju*

***Figure 1** Determination of the polygonal line that approximates the required integral curve*

3. PICARDOVA METODA

3. PICARD'S METHOD

A function $f\,(x,y)$ in the differential equation (1) on a closed scope P

$$|x - x_0| \le a, \qquad |y - y_0| \le b, \\ a > 0, b > 0$$

meets the requirements:

1° It is defined and uninterrupted, hence limited, i.e., there is such a real number $M > 0$ that it is true that

$$|f(x,y)| \le M,\ (x,y) \in P$$

2° A positive, real number L (Lipschitz constant) exists such that it holds true that

$$|f(x, y_2) - f(x, y_1)| \le L|y_2 - y_1|.$$

The differential equation (1) can be written in its integral form

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(\mathrm{x}))dx.$$

We define the series of functions $(y_n\ (x))$:

$$y_0(x) = y_0,\ y_n(x) = y_0 + \\ + \int_{x_0}^{x} f(x, y_{n-1}(\mathrm{x}))dx, \quad n = 1,2, \ldots . \qquad (3)$$

Provided that the requirements 1° and 2° have been met, then the series $(y_n\ (x))$ converges to the solution of the differential equation (1) on the interval $|\ x - x_0\ | \le h,\ h = min\left\{a, \frac{b}{M}\right\}$.

Example 1. Let's solve the differential equation $y' = 2x - y,\ y(0) = 1$

a) analytically b) using Euler's method

c) using Picard's method

Solution:

a) The differential equation can be note din the form of $y' + y = 2x$. It is a linear nonhomogeneous differential equation od the 1st order which we will solve by the constant variation method. The first step is to solve the pertaining homogenous $y' + y = 0$:

$$\frac{dy}{dx} = -y;\ \frac{dy}{y} = -dx;$$
$$\ln y = -x + C_1$$
$$y = e^{-x+C_1}$$
$$y = e^{C_1}e^{-x}$$

So, the final general solution of the pertaining homogenous equation is $y = Ke^{-x}$.

We search the general solution of the initial equation in the form of $y = K(x)e^{-x}$. By substituting into the differential equation, we will determine the unknown function $K(x)$:

$$K'(x)e^{-x} + K(x)e^{-x}(-1) + K(x)e^{-x} = 2x$$

$$K'(x)e^{-x} = 2x$$

$$K'(x) = 2xe^{x}$$

$$K(x) = 2\int x\,e^{x}dx$$

$$K(x) = \left[\begin{matrix} u = x, du = dx \\ dv = e^{x}dx, v = e^{x}\end{matrix}\right] =$$
$$= 2(xe^{x} - \int e^{x}dx =$$
$$= 2(xe^{x} - e^{x}) + C =$$
$$= 2e^{x}(x-1) + C$$

Thus, the general solution of the initial equation is

$$y = \mathrm{e}^{-\mathrm{x}}(2\mathrm{e}^{\mathrm{x}}(\mathrm{x}-1) + \mathrm{C})$$

i.e.

$$y = 2(x-1) + Ce^{-x}$$

To determine the particular solution, the initial condition must be applied $y(0) = 1$. By substituting $x = 0$, $y = 1$ in to the general solution, we obtain $1 = 2(0 - 1) + C \cdot e^{-0}$, and thus $C = 3$. Accordingly, the particular solution of the differential equation $y' = 2x - y$, $y(0) = 1$ is the function

$$y = 2(x-1) + 3e^{-x}.$$

b) For a better approximation, bigger n should be taken, which causes the increase of the number of operations that lead to the approximate numerical solution. In this example, let $c = 1$ and $n = 8$, then $h = \frac{1-0}{8} = 0.125$. According to the formula (2) we get:

Tablica 1 *primjer gdje su c = 1 i n = 8*

Table 1 *Example where c = 1 and n = 8*

x_i	y_i
$x_0 = 0.000$	$y_0 = 1.00000$
$x_1 = 0.125$	$y_1 = 0.87500$
$x_2 = 0.250$	$y_2 = 0.79688$
$x_3 = 0.375$	$y_3 = 0.75977$
$x_4 = 0.500$	$y_4 = 0.75977$
$x_5 = 0.625$	$y_5 = 0.78873$
$x_6 = 0.750$	$y_6 = 0.84639$
$x_7 = 0.875$	$y_7 = 0.92809$
$x_8 = 1.000$	$y_8 = 1.03083$

c) According to the formula (3) for $n = 4$, the first four approximations as per Picard's method are:

$$y_1 = 1 + \int_0^x (2x-1)dx = 1 - x + x^2$$
$$y_2 = 1 + \int_0^x (2x - (1 - x + x^2)dx =$$
$$= 1 - x + \frac{3}{2}x^2 - \frac{1}{3}x^3$$

$$y_3 = 1 + \int_0^x \left(2x - (1 - x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right) =$$

$$= 1 - x + \frac{3}{2}x^2 - \frac{1}{2} + \frac{1}{12}x^4$$

Finally, by way of y_3, and using the same process, we find

$$y_4 = 1 - x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{8}x^4 - \frac{1}{60}x^5$$

The approximate numerical solution obtained for the fourth approximation by using the Picard's method for the arguments' values obtained by dividing the interval [0,1] into eight equal parts is:

Tablica 2 *Približno numeričko rješenje dobiveno za četvrtu aproksimaciju po Picardovoj metodi*

Table 2 *Approximate Numerical Solution Obtained for the Fourth Approximation Using Picard's Method*

x_i	y_i - Picard
0.000	1.00000
0.125	0.89749
0.250	0.83641
0.375	0.81192
0.500	0.81979
0.625	0.85635
0.750	0.91841
0.875	1.00320
1.000	1.10833

To compare the accuracy of the numerical solution, the same values of the variable x on the interval [0,1] can be taken from the Euler's method, the fourth approximation using Picard's method and the exact analytical particular solution. After substituting and taking the y_i values from Table 1 and Table 2, the following is obtained:

Tablica 3 *Prikaz rezultata*

Table 3 *Presentation of Results*

x_i	y_i -Euler	y_i- Picard	y_i -točno rješenje
0.000	1.00000	1.00000	1.00000
0.125	0.87500	0.89749	0.89749
0.250	0.79688	0.83641	0.83640
0.375	0.75977	0.81192	0.81187
0.500	0.75854	0.81979	0.81959
0.625	0.78873	0.85635	0.85578
0.750	0.84639	0.91841	0.91710
0.875	0.92809	1.00320	1.00059
1.000	1.03083	1.10833	1.10364

The results shown in Table 3 demonstrate that Picard's method provides significantly more accurate values for $n = 4$ already, compared to Euler's method.

4. PRIMJENA SAGEMATH PROGRAMA

4. APPLICATION OF THE SAGEMATH PROGRAM

Approximate calculations generally require a significant amount of computation, substitution, and verification, especially when required high precision dictates numerous steps. This increases the likelihood of errors, not only in calculations but also in substitutions and transcribing previously obtained values. Modern technology applications greatly facilitate many aspects of this process without requiring expertise in programming. Below is a demonstration of how the previous example can be solved using the SageMath programme.

Example 2. Solving the differential equation $y' = 2x - y$, $y(0) = 1$ y applying SageMath software
a) analytically b) using Euler's method c) using Picard's method

a) To find the analytical solution, it is sufficient to write:

```
var('x')
y=function('y')(x)
d=diff(y,x)==2*x-y
y=desolve(d,[y,x],ics=[0,1])
y=expand(y)
print('y={0}'.format(y))
```

y=2*x + 3*e^(-x) – 2

The function's values for $x_0 = 0$, $x_1 = 0.25$, ..., $x_8 = 1$ can be found using the code:

```
var('x)
y=2*(x-1)+3*exp(-x)
for i in range (9):

print(0+i*0.125.n(digits=4),y(0+i*0.125).n(digits=6))
```

(0.0000, 1.00000)
(0.1250, 0.897491)
(0.2500, 0.836402)
(0.3750, 0.811868)
(0.5000, 0.819592)
(0.6250, 0.855784)
(0.7500, 0.917100)
(0.8750, 1.00059)
(1.000, 1.10364)

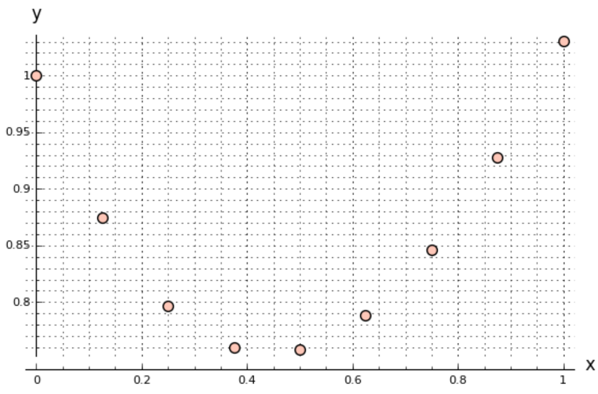
b) The programme that executes Euler's method and saves the points in a list for graphical representation.

```
var('x,y')
x=0;y=1;h=(1.-0)/8;tocke=[];
for k in range (9):
   tocke.append((x,y))
   print(x,y)
   f=(2*x-y).n(digits=6)
   x=(x+h).n(digits=6)
   y=(y+f*h).n(digits=6)
```

(0, 1)
(0.125000, 0.875000)
(0.250000, 0.796875)
(0.375000, 0.759766)
(0.500000, 0.758545)
(0.625000, 0.788727)
(0.750000, 0.846386)
(0.875000, 0.928088)
(1.00000, 1.03083)

Graphical representation (Figure 2.):

```
g = scatter_plot(tocke)
g.axes_labels(['x','y'])
g.fontsize(8)
g.show(gridlines=['minor','minor'])
```



***Slika 2** Grafički prikaz*

***Figure 2** Graphical representation*

c) The programme that finds the first four approximations using Picard's method and calculates values in y4 for $x_0 = 0$, $x_1 = 0.25$, ..., $x_8 = 1$.

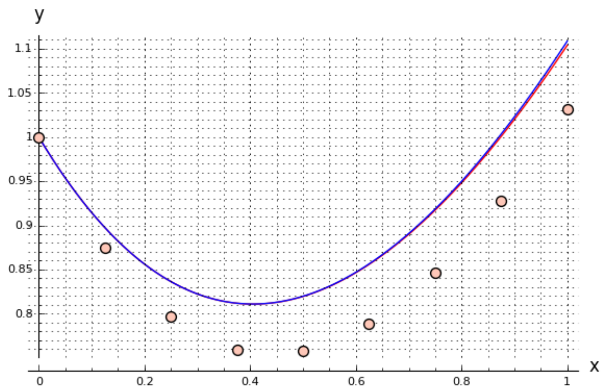
```
var('x')
y=1
for k in range (4):
   f=2*x-y
   y=1+integral(f,x,0,x)
   print('y={0}'.format(y))
for i in range (9):

print(0+i*0.125.n(digits=3),y(0+i*0.125).n(digits=6))
```

y=x^2 - x + 1
y=-1/3*x^3 + 3/2*x^2 - x + 1
y=1/12*x^4 - 1/2*x^3 + 3/2*x^2 - x + 1
y=-1/60*x^5 + 1/8*x^4 - 1/2*x^3 + 3/2*x^2 - x + 1
(0.000, 1.00000)
(0.125, 0.897491)
(0.250, 0.836410)
(0.375, 0.811919)
(0.500, 0.819792)
(0.625, 0.856351)
(0.750, 0.918408)
(0.875, 1.00320)
(1.000, 1.10833)

The quality of the approximations can be compared by sketching the diagrams of the exact solution, the solution obtained by using Euler's method and the fourth approximation from Picard's method in the same rectangular coordinate system (Figure 3.). This can be executed in the following way:

```
var('x')
g=scatter_plot(tocke)+plot(2*x + 3*e^(-x) - 2,(x,0,1),color='red')+plot(-1/60*x^5 + 1/8*x^4 - 1/2*x^3 + 3/2*x^2 - x + 1,(x,0,1))
g.axes_labels(['x','y'])
g.fontsize(8)
g.show(gridlines=['minor','minor'])
```



***Slika 3** Skicirani graf funkcije točnog rješenja*

***Figure 3** Sketched graph of the exact solution function*

5. ZAKLJUČAK
5. CONCLUSION

The application of modern technologies significantly impacts solving various problems in everyday life. Beyond posing challenges, technology enables simplification and easier resolution of issues across diverse fields of human activity. This work provides a comparative analysis of numerically solving first-order ordinary differential equations using Euler's and Picard's methods, both classically and through the SageMath software.

The primary objective is to first highlight the advantages of numerical solutions, considering that many differential equations cannot be solved via elementary analytical approaches, and to emphasize the utility of widely accessible programming tools. Additionally, the graphical interpretation of the obtained results is integrated to visually validate and compare the methods.

6. REFERENCE
6. REFERENCES

[1.] W. Boyce, R. DiPrima, Elementary Differential Equations and Boundary Value Problems, John Wiley & Sons, 11 th Edition 2017.

[2.] M. Rahman, Integral Equations and their Applications, WIT Press Southampton, 2007.

[3.] R. Scitovski, Numerička matematika, Odjel za matematiku Osijek, Osijek, 2015.

[4.] SageMath, https://www.sagemath.org/

AUTORI · *AUTHORS*

• **Edmond Krusha** - nepromijenjena biografija nalazi se u časopisu Polytechnic & Design Vol. 9, No. 2, 2021.

Korespondencija · ***Correspondence***
e.krusha@gmail.com

• **Marijan Čančarević** - diplomirao program studija Matematike na Prirodoslovno matematičkom fakultetu u Zagrebu. Voditelj je prijediplomskog studija i katedre za matematiku i fiziku na Veleučilištu suvremenih informacijskih tehnologija u Zagrebu (VSITE). Na VSITE -u nositelj i predavač za više kolegija iz područja prirodnih znanosti, polje matematika. Tijekom akademske karijere surađivao sa Splitskim Veleučilištem, Veleučilištem u Varaždinu, Veleučilištem u Velikoj Gorici i surađuje s Veleučilištem u Križevcima. Područje znanstvenog i stručnog interesa mu je primjena suvremenih informacijskih tehnologija u nastavi matematičkih kolegija, metodika nastave matematike i primijenjena matematika. Kao koautor i autor objavio više znanstvenih i stručnih radova. Koautor je udžbenika Matematika 1, autor recenziranih nastavnih materijala za kolegij Primijenjena i numerička matematika i laboratorijskih vježbi za matematičke kolegije koje predaje na VSITE-u.

Korespondencija · ***Correspondence***
marijan.cancarevic@vsite.hr

• **Ivica Čulina** - rođen je 1987. godine. Nakon završetka prirodoslovno matematičke gimnazije, 2011. diplomira elektrotehniku i informacijske tehnologije na zagrebačkom FER-u. Nakon rada na inžinjerskim i komercijalnim poslovima, 2022. zapošljava se na Veleučilištu Velika Gorica kao predavač. Područja stručnog rada i interesa: UNIX operacijski sustavi, Operacijski sustavi u realnom vremenu, Ugradbeni sustavi i Web tehnologije.

Korespondencija · ***Correspondence***
ivica.culina@vvg.hr